

SCHRÖDINGER OPERATORS AND THE DISTRIBUTION OF RESONANCES IN SECTORS

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ABSTRACT. The purpose of this paper is to give some refined results about the distribution of resonances in potential scattering. We use techniques and results from one and several complex variables, including properties of functions of completely regular growth. This enables us to find asymptotics for the distribution of resonances in sectors for certain potentials and for certain families of potentials.

1. INTRODUCTION

The purpose of this paper is to prove some results about the distribution of resonances in potential scattering. In particular, we study the distribution of resonances in sectors and give asymptotics of the “expected value” of the number of resonances in certain settings.

More precisely, we consider the operator $-\Delta + V$, where $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and Δ is the (non-positive) Laplacian. Then, except for a finite number of values of λ , $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$, $\text{Im } \lambda > 0$, is a bounded operator on $L^2(\mathbb{R}^d)$ for λ in the upper half plane. When d is odd and $\chi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ satisfies $\chi V = V$, $\chi R_V(\lambda) \chi$ has a meromorphic continuation to the lower half plane. The poles of $\chi R_V(\lambda) \chi$ are called *resonances*, and are independent of choice of χ satisfying these hypotheses. Resonances are analogous to eigenvalues not only in their appearance as poles of the resolvent, but also because they appear in trace formulas much as eigenvalues do [1, 9, 12]. Physically, they may be thought of as corresponding to decaying waves.

Let $n_V(r)$ denote the number of resonances of $-\Delta + V$, counted with multiplicity, with norm at most r . When $d = 1$, asymptotics of $n_V(r)$ are known:

$$\lim_{r \rightarrow \infty} \frac{n_V(r)}{r} = \frac{2}{\pi} \text{diam}(\text{supp}(V))$$

[19]; see also [5, 15, 17]. Moreover, “most” of the resonances occur in sectors about the real axis, in the sense that for any $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{\#\{\lambda_j \text{ pole of } R_V(\lambda) : |\arg \lambda_j - \pi| < \epsilon \text{ or } |\arg \lambda_j - 2\pi| < \epsilon\}}{r} = \frac{2}{\pi} \text{diam}(\text{supp}(V)),$$

[5]. These results are valid for complex-valued V . The case $d = 1$ is exceptional, though: in higher dimensions much less is known.

Now we turn to $d \geq 3$ odd, where the question is more subtle. If $V \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0, a) = \{x \in \mathbb{R}^d : |x| \leq a\}$, then

$$(1.1) \quad d \int_0^r \frac{n_V(t) - n_V(0)}{t} dt \leq c_d a^d r^d + o(r^d).$$

where c_d is defined in (3.5) and depends only on the dimension. Zworski [21] showed that such a bound holds, and Stefanov [18] identified the optimal constant. There are examples for which equality holds in (1.1), [20, 18]. Lower bounds have proved more elusive. The current best known general quantitative lower bound is for non-trivial real-valued $V \in C_c^\infty(\mathbb{R})$

$$(1.2) \quad \limsup_{r \rightarrow \infty} \frac{n_V(r)}{r} > 0$$

[16]. On the other hand, there are nontrivial complex-valued potentials V for which $\chi R_V(\lambda) \chi$ has no poles, [3].

We wish to single out the set for which asymptotics actually hold in (1.1). This is the set defined, for $a > 0$, as

$$(1.3) \quad \mathfrak{M}_a = \{V \in L^\infty(\mathbb{R}^d) : \text{supp } V \subset \overline{B}(0, a) \text{ and } n_V(r) = c_d a^d r^d + o(r^d) \text{ as } r \rightarrow \infty\}.$$

We remark that it is equivalent to require, as $r \rightarrow \infty$, $n_V(r) = c_d a^d r^d + o(r^d)$ or $d \int_0^r t^{-1} (n_V(t) - n_V(0)) dt = c_d a^d r^d + o(r^d)$. The set \mathfrak{M}_a contains infinitely many radial potentials. By results of [20, 18], this set contains any potential of the form $V(x) = v(|x|)$, where $v \in C^2([0, a])$ is real-valued, $v(a) \neq 0$, and $V(x) = 0$ for $|x| > a$. Additionally, it contains infinitely many complex-valued potentials which are isoresonant with these real-valued radial potentials [4].

We now can state some results. For the first, we set, for $\varphi < \theta$, $n_V(r, \varphi, \theta)$ to be the set of poles of $R_V(\lambda)$, counted with multiplicity, with norm at most r and with argument between φ and θ inclusive.

Proposition 1.1. *Let $V \in \mathfrak{M}_a$. Then, if $0 < \varphi < \theta < \pi$,*

$$n_V(r, \pi + \varphi, \pi + \theta) = \frac{1}{2\pi d} \tilde{s}_d(\varphi, \theta) r^d a^d + o(r^d) \text{ as } r \rightarrow \infty$$

where

$$\tilde{s}_d(\varphi, \theta) = h'_d(\theta) - h'_d(\varphi) + d^2 \int_\varphi^\theta h_d(s) ds,$$

and $h_d(\theta)$ is as defined in (3.4).

If V is real-valued, then λ_0 is a resonance of $-\Delta + V$ if and only if $-\overline{\lambda_0}$ is a resonance. In this case for $V \in \mathfrak{M}_a$ and $0 < \theta < \pi$

$$(1.4) \quad n_V(r, \pi, \pi + \theta) = \frac{1}{2\pi d} \left[h'_d(\theta) + d^2 \int_0^\theta h_d(s) ds \right] a^d r^d + o(r^d).$$

Here, as elsewhere in this paper, we are concerned with the behavior as $r \rightarrow \infty$. Thus, one should understand that statements of the type $f(r) = g(r) + o(r^p)$ are statements which hold for r sufficiently large.

Corollary 1.4 shows that (1.4) holds for any $V \in \mathfrak{M}_a$. These results show that any potential in \mathfrak{M}_a must have resonances distributed regularly in sectors, as well as being distributed regularly in balls centered at the origin. A result like this proposition and Corollary 1.4 is, for the special potentials of the form $V(x) = v(|x|)$ mentioned earlier, implicit in the papers of Zworski [20] and Stefanov [18]. Here we derive it as a corollary of some complex-analytic results, and note that it holds for *any* potential $V \in \mathfrak{M}_a$. We note that this proposition could, in fact, follow as a corollary to Theorem 1.3. However, we prefer to give a separate proof that uses standard results for functions of completely regular growth.

In the following theorem we use the notation $N_V(r) = \int_0^r \frac{1}{t} (n_V(t) - n_V(0)) dt$. This theorem shows that there are many potentials for which something close to the optimal upper bound on the resonances is achieved.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^p$ be an open connected set. Suppose $V(z) = V(z, x)$ is holomorphic in $z \in \Omega$ and, for each $z \in \Omega$, $V(z, x) \in L^\infty(\mathbb{R}^d)$, and $V(z, x) = 0$ if $|x| > a$. Suppose in addition that for some $z_0 \in \Omega$, $V(z_0) \in \mathfrak{M}_a$. Then there is a pluripolar set $E \subset \Omega$ so that*

$$\limsup_{r \rightarrow \infty} \frac{N_{V(z)}(r)}{r^d} = \frac{c_d a^d}{d} \text{ for all } z \in \Omega \setminus E.$$

Moreover, for any $\theta > 0$, $\theta < \pi$, there is a pluripolar set E_θ so that

$$\limsup_{r \rightarrow \infty} \frac{N_{V(z)}(r, \pi, \pi + \theta)}{r^d} \geq \lim_{\epsilon \downarrow 0} \frac{a^d}{4\pi d^2} h'_d(\epsilon)$$

for all $z \in \Omega \setminus E_\theta$.

For example, one may take, for $z \in \mathbb{C}$, $V(z) = zV_1 + (1 - z)V_0$, where $V_0 \in \mathfrak{M}_a$ and $V_1 \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0, a)$. Since $h'_d(0+) = \lim_{\epsilon \downarrow 0} h'_d(\epsilon) > 0$, see Lemma 3.3, the second statement of the theorem is meaningful. This result is of particular interest since resonances near the real axis are considered the more physically relevant ones.

We recall the definition of a pluripolar set in Section 2. Here we mention that a pluripolar set is small. A pluripolar set $E \subset \mathbb{C}^p$ has \mathbb{R}^{2p} Lebesgue measure 0, and if $E \subset \mathbb{C}$ is pluripolar, $E \cap \mathbb{R}$ has one-dimensional Lebesgue measure 0 (see, for example, [10, 14]). Thus the statements of Theorem 1.2 hold for “most” values of $z \in \Omega$.

If we take a weighted average over a family of potentials, a kind of expected value, we are able to find asymptotics analogous to those which hold for a potential in \mathfrak{M}_a .

In the statement of the next theorem and later in the paper, we use the notation $d\mathcal{L}(z) = d\operatorname{Re} z_1 d\operatorname{Im} z_1 \cdots d\operatorname{Re} z_p d\operatorname{Im} z_p$.

Theorem 1.3. *Suppose the hypotheses of Theorem 1.2 are satisfied. Then for any $\psi \in C_c(\Omega)$,*

$$\int_{\Omega} \psi(z) n_{V(z)}(r) d\mathcal{L}(z) = c_d a^d r^d \int_{\Omega} \psi(z) d\mathcal{L}(z) + o(r^d)$$

as $r \rightarrow \infty$. Additionally, we have, for $0 < \varphi < \theta < \pi$,

$$\int_{\Omega} \psi(z) n_{V(z)}(r, \varphi + \pi, \theta + \pi) d\mathcal{L}(z) = \frac{1}{2\pi d} \tilde{s}_d(\varphi, \theta) r^d a^d \int_{\Omega} \psi(z) d\mathcal{L}(z) + o(r^d)$$

where \tilde{s}_d is as defined in Proposition 1.1. Moreover, for $0 < \theta < \pi$,

$$\begin{aligned} \int_{\Omega} \psi(z) n_{V(z)}(r, \pi, \theta + \pi) d\mathcal{L}(z) \\ = \frac{1}{2\pi d} \left[h'_d(\theta) + d^2 \int_0^{\theta} h_d(s) ds \right] a^d r^d \int_{\Omega} \psi(z) d\mathcal{L}(z) + o(r^d). \end{aligned}$$

Corollary 1.4. *Let $V \in \mathfrak{M}_a$. For any $0 < \theta < \pi$,*

$$(1.5) \quad n_V(r, \pi, \theta + \pi) = \frac{1}{2\pi d} \left[h'_d(\theta) + d^2 \int_0^{\theta} h_d(s) ds \right] a^d r^d + o(r^d)$$

and, for any $0 < \varphi < \pi$,

$$(1.6) \quad n_V(r, \varphi + \pi, 2\pi) = \frac{1}{2\pi d} \left[-h'_d(\varphi) + d^2 \int_{\varphi}^{\pi} h_d(s) ds \right] a^d r^d + o(r^d)$$

as $r \rightarrow \infty$.

This corollary follows from Theorem 1.3 by taking $V(z)$ equal to the constant (in z) potential V . We could instead give a more direct proof by, essentially, simplifying the proof of Proposition 5.3 and then applying Lemma 5.4.

It is worth noting that the coefficients of r^d in (1.5) and (1.6) are positive, so that in any sector in the lower half plane which touches the real axis, the number of resonances grows like r^d .

The proofs of the results here are possible because of the optimal upper bounds on $\limsup_{r \rightarrow \infty} r^{-d} \ln |\det S_V(re^{i\theta})|$, $0 < \theta < \pi$, proved in [18], see Theorem 3.2 here. These, combined with some one-dimensional complex analysis, are used to prove Proposition 1.1, and could be used to give a direct proof of Corollary 1.4. The proofs of the other theorems use, in addition to one-dimensional complex analysis, some facts about plurisubharmonic functions. Many of the complex-analytic results which we shall use are recalled in Section 2.

Again, we emphasize that we are concerned here with large r behavior of resonance counting functions, and consequently of other functions as well. Thus, statements of the type $f(r) = g(r) + o(r^p)$ are to be understood as holding for large values of r .

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2. SOME COMPLEX ANALYSIS

In this section we recall some definitions and results from complex analysis in one and several variables. We will mostly follow the notation and conventions of [11] and [10]. We also prove a result, Proposition 2.2, for which we are unaware of a proof in the literature.

The *upper relative measure* of a set $E \subset \mathbb{R}_+$ is

$$\limsup_{r \rightarrow \infty} \frac{\text{meas}(E \cap (0, r))}{r}.$$

A set $E \subset \mathbb{R}_+$ is said to have *zero relative measure* if it has upper relative measure 0.

If f is a function holomorphic in the sector $\varphi < \arg z < \theta$, we shall say f is of order ρ if $\limsup_{r \rightarrow \infty} \frac{\ln \ln(\max_{\varphi < \phi < \theta} |f(re^{i\phi})|)}{\ln r} = \rho$. We shall further restrict ourselves to functions of order ρ and *finite type*, so that

$$\limsup_{r \rightarrow \infty} \frac{\ln(\max_{\varphi < \phi < \theta} |f(re^{i\phi})|)}{r^\rho} < \infty.$$

We shall use similar definitions for a function holomorphic in a neighborhood of a closed sector. In this section only, we shall, following Levin [11], use the notation h_f for the *indicator function* (or *indicator*) of a function f of order ρ :

$$h_f(\theta) \stackrel{\text{def}}{=} \limsup_{r \rightarrow \infty} (r^{-\rho} \ln |f(re^{i\theta})|).$$

Suppose f is a function analytic in the angle (θ_1, θ_2) and of order ρ and finite type there. The function f is of *completely regular growth* on some set of rays $R_{\mathfrak{M}}$ (\mathfrak{M} is the set of values of θ) if the function

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} \frac{\ln |f(re^{i\theta})|}{r^\rho}$$

converges uniformly to $h_f(\theta)$ for $\theta \in \mathfrak{M}$ when r goes to infinity taking on all positive values except possibly for a set $E_{\mathfrak{M}}$ of zero relative measure. The function f is of *completely regular growth in the angle* (θ_1, θ_2) if it is of completely regular growth on every closed interior angle.

Functions of completely regular growth have zeros that are rather regularly distributed. For a function f holomorphic in $\{z : \theta_1 < \arg z < \theta_2\}$ we define, for

$\theta_1 < \varphi < \theta < \theta_2$, $m_f(r, \varphi, \theta)$ to be the number of zeros of $f(z)$ in the sector $\varphi \leq \arg z \leq \theta$, $|z| \leq r$.¹ We recall the following theorem from [11].

Theorem 2.1. [11, Chapter III, Theorem 3] *If a holomorphic function $f(z)$ of order d and finite type has completely regular growth within an angle (θ_1, θ_2) , then for all values of φ and θ , $(\theta_1 < \varphi < \theta < \theta_2)$ except possibly for a denumerable set, the following limit will exist:*

$$\lim_{r \rightarrow \infty} \frac{m_f(r, \varphi, \theta)}{r^d} = \frac{1}{2\pi d} \tilde{s}_f(\varphi, \theta)$$

where

$$\tilde{s}_f(\varphi, \theta) = \left[h'_f(\theta) - h'_f(\varphi) + d^2 \int_{\varphi}^{\theta} h_f(s) ds \right].$$

The exceptional denumerable set can only consist of points for which $h'_f(\theta + 0) \neq h'_f(\theta - 0)$.

In the following proposition we use the notation $m_f(r)$ to denote the number of zeros of a function f , counted with multiplicity, with norm at most r . It is likely that some of the hypotheses included here could be relaxed. However, when we apply this proposition, f will be the determinant of the scattering matrix, perhaps multiplied by a rational function, and many of these hypotheses are natural in such applications.

Let $f(z)$ be a function meromorphic on \mathbb{C} . Then $f(z) = g_1(z)/g_2(z)$, with g_1, g_2 entire. The functions g_1 and g_2 are not uniquely determined. However, the order of f can be defined to be

$$\min\{\max(\text{order of } g_1, \text{order of } g_2) : f(z) = g_1(z)/g_2(z) \text{ with } g_1, g_2 \text{ entire}\}.$$

It is possible to define the order of a meromorphic function by using the Nevanlinna characteristic function, yielding the same result.

Proposition 2.2. *Let f be a function meromorphic in the complex plane, with neither zeros nor poles on the real line. Suppose all the zeros of f lie in the open upper half plane, and all the poles in the open lower half plane. Furthermore, assume f is of order $d > 1$, h_f is finite for $0 \leq \theta \leq \pi$, and $h_f(\theta_0) \neq 0$ for some θ_0 , $0 < \theta_0 < \pi$. Suppose in addition*

$$(2.1) \quad \int_0^r \frac{f'(t)}{f(t)} dt = o(r^d) \text{ as } r \rightarrow \pm\infty,$$

and the number of poles of f with norm at most r is of order at most d . If

$$\liminf_{r \rightarrow \infty} \frac{m_f(r)}{r^d} = \frac{d}{2\pi} \int_0^\pi h_f(\theta) d\theta,$$

then f is of completely regular growth in the angle $(0, \pi)$.

¹More standard notation would be $n(r, \varphi, \theta)$, but we have already defined $n_V(r, \varphi, \theta)$ to be something else.

Before proving the proposition, we note that Govorov [7, 8] has studied the issue of completely regular growth of functions holomorphic in an angle. This is discussed in [11, Appendix VIII, section 2]. This is somewhat different than what we consider, since we use the assumption that f is meromorphic and of order d on the plane. Thus Govorov uses different restrictions on the distribution of the zeros of f .

Proof. The proof of this proposition follows in outline the proof of the analogous theorem for entire functions in the plane, [11, Chapter IV, Theorem 3]. Rather than using Jensen's theorem, though, it uses the equality

$$(2.2) \quad \int_0^r \frac{m_f(t)}{t} dt = \frac{1}{2\pi} \operatorname{Im} \int_0^r \frac{1}{t} \int_{-t}^t \frac{f'(s)}{f(s)} ds dt + \frac{1}{2\pi} \int_0^\pi \ln |f(re^{i\theta})| d\theta$$

if $|f(0)| = 1$, which follows using the proof of [6, Lemma 6.1].

By [11, Property (4), Chapter I, section 12],

$$(2.3) \quad \liminf_{r \rightarrow \infty} \frac{m_f(r)}{r^d} \leq \liminf_{r \rightarrow \infty} dr^{-d} \int_0^r \frac{m_f(t)}{t} dt.$$

We note [11, Chapter I, Theorem 28] that for any $\epsilon > 0$ there is an $R > 0$ so that

$$(2.4) \quad r^{-d} \ln |f(re^{i\theta})| \leq h_f(\theta) + \epsilon, \text{ for } r > R, 0 \leq \theta \leq \pi.$$

Using this, (2.2), and our assumptions on the behavior of f on the real axis, we see that

$$\limsup_{r \rightarrow \infty} r^{-d} \int_0^r \frac{m_f(t)}{t} dt \leq \frac{1}{2\pi} \int_0^\pi h_f(\theta) d\theta.$$

Combining this with (2.3) and using our assumptions on $m_f(r)$, we get

$$\lim_{r \rightarrow \infty} r^{-d} \int_0^r \frac{m_f(t)}{t} dt = \frac{1}{2\pi} \int_0^\pi h_f(\theta) d\theta.$$

Thus using (2.2) and (2.1) again, we have

$$\lim_{r \rightarrow \infty} \int_0^\pi [h_f(\theta) - r^{-d} \ln |f(re^{i\theta})|] d\theta = 0,$$

and, using (2.4),

$$\lim_{r \rightarrow \infty} \int_0^\pi |h_f(\theta) - r^{-d} \ln |f(re^{i\theta})|| d\theta = 0.$$

Since we have assumed f is of order d , we may write f as the quotient of two entire functions, each of order at most d . Then we may apply [11, Chapter 2, Theorem 7] to find that for every $\eta > 0$ there is a set E_η of positive numbers of upper relative measure less than η so that if $r \notin E_\eta$, the family of functions of θ ,

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} r^{-d} \ln |f(re^{i\theta})|,$$

is equicontinuous in the angle $0 < \epsilon_0 \leq \theta \leq \pi - \epsilon_0$.

Given $\eta > 0$ and $\epsilon > 0$ we can, by the above result, find a $\delta > 0$ with $(\theta_1 - \delta, \theta_2 + \delta) \subset (0, \pi)$ and a set E_η of upper relative measure at most η so that if $\theta \in (\theta_1, \theta_2)$, $r \notin E_\eta$,

and $|\varphi - \theta| < \delta$, then $|h_{f,r}(\theta) - h_{f,r}(\varphi)| < \epsilon/4$ and $|h_f(\theta) - h_f(\varphi)| < \epsilon/4$. Then for $0 < |k| < \delta$, $r \notin E_\eta$,

$$\begin{aligned} |h_{f,r}(\theta) - h_f(\theta)| &< \epsilon/2 + \frac{1}{k} \int_{\theta}^{\theta+k} |h_{f,r}(\varphi) - h_f(\varphi)| d\varphi \\ &\leq \epsilon/2 + \frac{1}{k} \int_0^\pi |h_{f,r}(\varphi) - h_f(\varphi)| d\varphi. \end{aligned}$$

Since the integral goes to 0 as $r \rightarrow \infty$, we have shown that for $r > r_\epsilon$, $r \notin E_\eta$, $|h_{f,r}(\theta) - h_f(\theta)| < \epsilon$. Since $\eta > 0$ and $\epsilon > 0$ are arbitrary, we have, by [11, Chapter III, Lemma 1], f is of completely regular growth in (θ_1, θ_2) . \square

We shall also need some basics about plurisubharmonic functions and pluripolar sets. We use notation as in [10] and refer the reader to this reference for more details.

Let $\Omega \subset \mathbb{C}^p$ be an open connected set. A function $\Psi : \Omega \rightarrow [-\infty, \infty)$ is said to be *plurisubharmonic* if $\Psi \not\equiv -\infty$, Ψ is upper semi-continuous, and

$$\Psi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \Psi(z + wre^{i\theta}) d\theta$$

for all w, r such that $z + uw \subset \Omega$ for all $u \in \mathbb{C}$, $|u| \leq r$. The classic example of a plurisubharmonic function is $\ln |f(z)|$, where $f(z)$ is holomorphic. A subset $E \subset \Omega \subset \mathbb{C}^p$ is said to be *pluripolar* if there is a function Ψ plurisubharmonic on Ω so that $E \subset \{z : \psi(z) = -\infty\}$.

For the convenience of the reader, we recall [10, Proposition 1.39], which is the main additional fact from several complex variables which we shall need.

Proposition 2.3. ([10, Prop. 1.39]) *Let $\{\Psi_q\}$ be a sequence of plurisubharmonic functions uniformly bounded above on compact subsets in an open connected set $\Omega \subset \mathbb{C}^p$, with $\limsup_{q \rightarrow \infty} \Psi_q \leq 0$ and suppose that there exist $\xi \in \Omega$ such that $\limsup_{q \rightarrow \infty} \Psi_q(\xi) = 0$. Then $A = \{z \in \Omega : \limsup_{q \rightarrow \infty} \Psi_q(z) < 0\}$ is pluripolar in Ω .*

3. THE FUNCTIONS $s_V(\lambda) = \det S_V(\lambda)$ AND $h_d(\theta)$

For $V \in L_{\text{comp}}^\infty(\mathbb{R}^d)$ and $\chi \in L_{\text{comp}}^\infty(\mathbb{R}^d)$ with $\chi V = V$, we have $\chi R_V(\lambda) \chi = \chi R_0(\lambda) \chi (I + V R_0(\lambda) \chi)^{-1}$. Since for any χ with compact support in \mathbb{R}^d , $\|\chi R_0(\lambda) \chi\| \leq c_\chi / |\lambda|$ when $\text{Im } \lambda \geq 0$, we see that $R_V(\lambda)$ can have only finitely many poles in the closed upper half plane.

For $V \in L_{\text{comp}}^\infty(\mathbb{R}^d)$, let $S_V(\lambda)$ be the associated scattering matrix and $s_V(\lambda) = \det S_V(\lambda)$. With at most finitely many exceptions, the poles of $s_V(\lambda)$ coincide with the poles of $R_V(\lambda)$, and the multiplicities agree. Moreover, $s_V(\lambda) s_V(-\lambda) = 1$. We recall [2, Lemma 3.1]:

Lemma 3.1. *Let $V \in L_{\text{comp}}^\infty(\mathbb{R}^d; \mathbb{C})$. For $\lambda \in \mathbb{R}$, there is a C_V so that*

$$\left| \frac{d}{d\lambda} \ln s_V(\lambda) \right| \leq C_V |\lambda|^{d-2}$$

whenever $|\lambda|$ is sufficiently large.

In fact, if $\text{supp } V \subset \overline{B}(0, a)$ there is a constant $\alpha_d = \alpha_{d,a}$ so that it suffices to take $|\lambda| \geq 2\alpha_d \|V\|_\infty$ for such a bound to hold. We note that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2\alpha_d \|V\|_\infty$ under these same assumptions on V ,

$$(3.1) \quad \|S_V(\lambda) - I\| \leq C|\lambda|^{-1}.$$

This is relatively easy to see from an explicit representation of the scattering matrix; see, for example, the proof of [2, Lemma 3.1]. The constants in the statement of [2, Lemma 3.1] and in (3.1) can be chosen to depend only on the dimension, $\|V\|_\infty$ and the support of V . We note that it follows from Lemma 3.1, (3.1), and (2.2) that as $r \rightarrow \infty$

$$(3.2) \quad \int_0^r \frac{n_V(t)}{t} dt = \int_0^\pi \ln |\det S_V(re^{i\theta})| d\theta + O(r^{d-1}).$$

Let

$$(3.3) \quad \rho(z) \stackrel{\text{def}}{=} \ln \frac{1 + \sqrt{1 - z^2}}{z} - \sqrt{1 - z^2}, \quad 0 < \arg z < \pi.$$

This is a function which arises in studying the asymptotics of Bessel functions; see [13]. To define the square root which appears here, take the branch cut on the negative real axis and define ρ to be a continuous function in $\{0 < \arg z < \pi\} \cup (0, 1)$ and use the principal branches of the logarithm and the square root when $z \in (0, 1)$.

We use some notation of [18]. Set, for $0 < \theta < \pi$,

$$(3.4) \quad h_d(\theta) \stackrel{\text{def}}{=} \frac{4}{(d-2)!} \int_0^\infty \frac{[-\text{Re } \rho]_+(te^{i\theta})}{t^{d+1}} dt$$

and set $h_d(0) = 0$, $h_d(\pi) = 0$. Now set

$$(3.5) \quad c_d \stackrel{\text{def}}{=} \frac{d}{2\pi} \int_0^\pi h_d(\theta) d\theta = \frac{2d}{\pi(d-2)!} \int_{\text{Im } z > 0} \frac{[-\text{Re } \rho]_+(z)}{|z|^{d+2}} dx dy.$$

This is the constant c_d which appears in (1.1).

We recall the following result of [18, Theorem 5], which we paraphrase to suit our setting; [18, Theorem 5] actually covers a much larger class of operators.

Theorem 3.2. *(from [18, Theorem 5]) Let $V \in L^\infty(\mathbb{R}^d)$ be supported in $\overline{B}(0, a)$.*

(a) For any $\theta \in [0, \pi]$,

$$(3.6) \quad \ln |s_V(re^{i\theta})| \leq h_d(\theta) a^d r^d + o(r^d) \text{ as } r \rightarrow \infty,$$

and the remainder term depends on V , and is uniform for $0 < \delta \leq \theta \leq \pi - \delta$ for any $\delta \in (0, \pi)$.

(b) For any $\delta > 0$,

$$\ln |s_V(re^{i\theta})| \leq (h_d(\theta)a^d + \delta)r^d + o(r^d) \text{ as } r \rightarrow \infty$$

uniformly in $\theta \in [0, \pi]$.

We remark that both of these statements are about “large r ” behavior, so that the possibility that s_V has a finite number of poles in the upper half plane does not affect the validity of the statements.

It is important to note several things about the bounds in this theorem. One is that although Stefanov’s theorem is stated only for self-adjoint operators (hence V real) it is equally valid when we allow complex-valued potentials. In fact, the proof of (a) in [18, Theorem 5] uses self-adjointness only to obtain a bound on the resolvent for λ in the upper half plane. A similar bound is true for the operator $-\Delta + V$ when V is complex-valued. The proof of (b) uses the fact that for real V , if $\lambda \in \mathbb{R}$, $\ln |s_V(\lambda)| = 1$. For complex-valued V , the proof in [18] of (b) can be adapted by using (3.1) and Lemma 3.1 to show that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2\alpha_d \|V\|_\infty$, $|\ln s_V(\lambda)| \leq C(1 + |\lambda|)^{d-1}$. Here C can be chosen to depend only on d , $\|V\|_\infty$ and the diameter of the support of V .

Likewise, the particulars of the operator enter only through the diameter of the support of the perturbation (for us, the diameter of the support of V , which is $2a$) and the aforementioned bound on the resolvent in the good half plane $\text{Im } \lambda > 0$. Thus, it is easy to see that the estimates of Theorem 3.2 are uniform in V as long as $\text{supp } V \subset \overline{B}(0, a)$, $\|V\|_\infty \leq M$, and $r \geq 2\alpha_d M$.

We note that the upper bound (1.1) on the integrated resonance-counting function holds with the constant c_d defined in (3.5) even if V is complex-valued. This follows from the proof in [18]. In fact, the proof uses the bounds recalled in Theorem 3.2 and the identity (2.2). Together with the bounds in Lemma 3.1 and (3.1), these prove (1.1), even when V is complex-valued.

We shall want to understand the function $h_d(\theta)$ better. Note that for $0 < \theta \leq \pi/2$,

$$h_d(\pi/2 + \theta) = h_d(\pi/2 - \theta).$$

This can be seen directly using the definition of h_d and ρ .

Lemma 3.3. *The function $h_d(\theta)$, defined in (3.4), is C^1 on $(0, \pi)$. Moreover,*

$$h'_d(0+) \stackrel{\text{def}}{=} \lim_{\epsilon \downarrow 0} h'_d(\epsilon) = \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{(d-2)! \Gamma\left(1 + \frac{d}{2}\right)}.$$

Proof. We note [13, Section 4] that $\text{Re } \rho(z) < 0$ if $0 < \arg z < \pi$ and $|z| > |z_0(\arg z)|$, where $z_0(\theta)$ is the unique point in \mathbb{C} with argument θ and which lies on the curve

given by

$$\pm(s \coth s - s^2)^{1/2} + i(s^2 - s \tanh s)^{1/2}, 0 \leq s \leq s_0.$$

Here s_0 is the positive solution of $\coth s = s$. Furthermore, $\operatorname{Re} \rho(z) > 0$ if z is in the upper half plane but $|z| < |z_0(\arg z)|$. Hence, recalling the definition of h_d , we have

$$h_d(\theta) = \frac{4}{(d-2)!} \int_{|z_0(\theta)|}^{\infty} \frac{[-\operatorname{Re} \rho](te^{i\theta})}{t^{d+1}} dt.$$

Using the definition of ρ (3.3) and the following comments, we see that ρ is in fact a smooth function of z with $0 < \arg z < \pi$, $|z| > 0$. Since $|\rho(z)|/|z| \rightarrow 1$ when $|z| \rightarrow \infty$ in this region, the integral defining h_d is absolutely convergent. Likewise, since

$$\frac{\partial}{\partial \theta} \rho(te^{i\theta}) = -i\sqrt{1 - (te^{i\theta})^2}$$

we have

$$\left| \frac{-\operatorname{Re} \left[\frac{\partial}{\partial \theta} \rho(te^{i\theta}) \right]}{t^{d+1}} \right| \leq Ct^{-d}$$

and the integral

$$\int_{|z_0(\theta)|}^{\infty} \frac{-\operatorname{Re} \left[\frac{\partial}{\partial \theta} \rho(te^{i\theta}) \right]}{t^{d+1}} dt$$

converges absolutely. A computation shows that $|z_0|$ is a C^1 function of θ for θ in $(0, \pi)$, and $\lim_{\epsilon \downarrow 0} \frac{\partial}{\partial \theta} |z_0|$ is finite. Thus, using that $\operatorname{Re} \rho(z_0(\theta)) = 0$ and the regularity of the derivative of $|z_0|(\theta)$, we get

$$\frac{d}{d\theta} h_d(\theta) = \frac{4}{(d-2)!} \int_{|z_0(\theta)|}^{\infty} \frac{\operatorname{Re} i\sqrt{1 - (te^{i\theta})^2}}{t^{d+1}} dt$$

which is continuous in θ . Thus h_d is C^1 on $(0, \pi)$, $h'_d(0+) = \frac{4}{(d-2)!} \int_1^{\infty} \frac{\sqrt{t^2-1}}{t^{d+1}} dt$, and a computation now finishes the proof of the lemma. \square

If $d = 3$, we can compute that

$$h_3(\theta) = \frac{4}{9} \left(\sin(3\theta) + \operatorname{Re} \frac{(1 - z_0^2(\theta))^{3/2}}{|z_0(\theta)|^3} \right)$$

where $z_0(\theta)$ is as in the proof of the lemma. We comment that the $\sin(3\theta)$ term is missing from the first remark following the statement of [18, Theorem 5].

4. PROOF OF PROPOSITION 1.1

We can now give the proof of Proposition 1.1, which follows by combining Theorem 2.1, Proposition 2.2, and [18, Theorem 5].

Recall that $S_V(\lambda)$ is the scattering matrix associated with the operator $-\Delta + V$, and $s_V(\lambda) = \det S_V(\lambda)$. Then s_V has a pole at λ if and only if s_V has a zero at $-\lambda$, and the multiplicities coincide. Moreover, with at most a finite number of exceptions, the poles of $s_V(\lambda)$ coincide, with multiplicity, with the zeros of $R_V(\lambda)$.

If $s_V(\lambda)$ has poles in the closed upper half plane, it has only finitely many, say $\lambda_1, \dots, \lambda_m$, where the poles are repeated according to multiplicity. Set

$$f(\lambda) = \prod_{j=1}^m \frac{(\lambda - \lambda_j)}{\lambda + \lambda_j} s_V(\lambda).$$

We check that f satisfies the hypotheses of Proposition 2.2. Note that f and $s_V(\lambda)$ have the same order and they have the same indicator function for $0 \leq \theta \leq \pi$. We know that s_V has order at most d by [22, Theorem 7]. Moreover, for any M chosen large enough that s_V has no zeros or poles bigger than M on the real line, for $r > M$ we have

$$\int_0^r \frac{f'(t)}{f(t)} dt = \int_M^r \frac{s'_V(t)}{s_V(t)} dt + O(1).$$

Using (3.1) and Lemma 3.1, we see that

$$\int_M^r \frac{s'_V(t)}{s_V(t)} dt = O(r^{d-1}) \text{ as } r \rightarrow \infty$$

yielding

$$(4.1) \quad \int_0^r \frac{f'(t)}{f(t)} dt = O(r^{d-1}) \text{ as } r \rightarrow \infty.$$

A similar argument gives the same bound for $r \rightarrow -\infty$. It remains to check the hypotheses on the indicator function; this is done in the next paragraph.

From [18, Theorem 5], recalled here in Theorem 3.2, for $0 \leq \theta \leq \pi$ and large r ,

$$r^{-d} \ln |f(re^{i\theta})| \leq a^d h_d(\theta) + o(1)$$

where we have some uniformity in θ — see Theorem 3.2. Thus, using the equation (2.2) and (4.1)

$$\limsup_{r \rightarrow \infty} r^{-d} N_V(r) = \limsup_{r \rightarrow \infty} r^{-d} \frac{1}{2\pi} \int_0^\pi \ln |f(re^{i\theta})| d\theta \leq \frac{a^d}{2\pi} \int_0^\pi h_d(\theta) d\theta.$$

But since $V \in \mathfrak{M}_a$,

$$\lim_{r \rightarrow \infty} r^{-d} N_V(r) = \frac{c_d a^d}{d} = \frac{a^d}{2\pi} \int_0^\pi h_d(\theta) d\theta$$

and we see that we must have

$$\limsup_{r \rightarrow \infty} r^{-d} \ln |f(re^{i\theta})| = a^d h_d(\theta) \text{ for almost every } \theta \in (0, \pi).$$

The left hand side of the above equation is the value of the indicator function of f at θ . But the indicator function of f is continuous on $(0, \pi)$ [11, Section 16, point a, page 54], and so is $h_d(\theta)$. Thus we must have

$$\limsup_{r \rightarrow \infty} r^{-d} \ln |f(re^{i\theta})| = a^d h_d(\theta) \text{ for } \theta \in (0, \pi).$$

Applying Proposition 2.2 to $f(\lambda)$, we see that $f(\lambda)$ is a function of completely regular growth in the upper half plane. Since $h_d(\theta)$ is a C^1 function of θ for $\theta \in (0, \pi)$, we get the proposition from Theorem 2.1.

5. PROOF OF THEOREM 1.3

This section proves Theorem 1.3. We begin by outlining the strategy of the proof.

For $0 < \varphi < \theta < 2\pi$, recall the notation $n_V(r, \varphi, \theta)$ for the number of poles of $R_V(\lambda)$ in the sector $\{z : |z| \leq r, \varphi \leq \arg z \leq \theta\}$. A representative claim of the theorem is that with $V(z)$, Ω as in the statement of the theorem, $0 < \theta < \pi$,

$$(5.1) \quad \int_{\Omega} \psi(z) n_{V(z)}(r, \pi, \theta + \pi) d\mathcal{L}(z) \\ = \frac{1}{2\pi d} \left[h'_d(\theta) + d^2 \int_0^\theta h_d(s) ds \right] a^d r^d \int_{\Omega} \psi(z) d\mathcal{L}(z) + o(r^d)$$

as $r \rightarrow \infty$ for any $\psi \in C_c(\Omega)$. We prove this via the intermediate step of showing that (5.1) holds for ψ which is the characteristic function of any suitable ball in Ω , Proposition 5.7. To get (5.1) for $\psi \in C_c(\Omega)$, we cover the support of ψ with the union of a finite number of small disjoint balls and a set of small volume. On each small ball we can approximate ψ by its value at the center of the ball and apply Proposition 5.7. This and the necessary estimates are done in the proof of the theorem which ends this section.

The proof of Proposition 5.7 is done in a number of steps. We set

$$N_V(r, \varphi, \theta) = \int_0^r \frac{1}{t} (n_V(t, \varphi, \theta) - n_V(0, \varphi, \theta)) dt.$$

Lemma 5.2 gives $\int_0^\theta N_V(r, \pi, \theta' + \pi) d\theta'$ as a sum of two integrals involving $\ln |s_V|$ and an error of order r^{d-1} . This follows from an application of one-dimensional complex analysis, Lemma 3.1 and (3.1). Next we consider the function

$$\Psi(z, r, \rho) \stackrel{\text{def}}{=} \frac{1}{\text{vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_0^\theta N_{V(z')}(r, \pi, \theta' + \pi) d\theta' d\mathcal{L}(z').$$

Notice that here we are averaging over balls of varying center z . Fix ρ small, and consider this as a function of z and r . Lemma 5.2 is used to show that Ψ is the sum of a function Ψ_1 which is plurisubharmonic in z and a function which is $O(r^{d-1})$. The proof of Proposition 5.3 uses a combination of properties of plurisubharmonic functions and the fact that $r^{-d} N_{V(z')}(r, \pi, \theta' + \pi)$ is not negative and can be (locally) uniformly bounded above for large r to prove an “averaged” in θ and r version of (5.1) for ψ the characteristic function of a ball in Ω satisfying some conditions. Propositions 5.5 and then 5.7 eliminate the need to average in θ and r , using Lemma 5.4.

The proofs of the other claims of Theorem 1.3 are quite similar; the proofs of Proposition 5.6 and the final proof of the theorem indicate the differences.

Now we turn to proving the theorem. We shall need an identity related to both (2.2) and to what Levin calls a generalized formula of Jensen [11, Chapter 3, section 2]. We define, following [11], for a function f meromorphic in a neighborhood of $\arg z = \theta$ and with $|f(0)| = 1$,

$$(5.2) \quad J_f^r(\theta) \stackrel{\text{def}}{=} \int_0^r \frac{\ln |f(te^{i\theta})|}{t} dt.$$

This integral is well-defined even if f has a zero or pole with argument θ .

Lemma 5.1. *Let f be holomorphic in $\varphi \leq \arg z \leq \theta$, $|f(0)| = 1$, f have no zeros with argument φ or θ and with norm less than r , and let $m(r, \varphi, \theta)$ be the number of zeros of f in the sector $\varphi < \arg z < \theta$, $|z| \leq r$. Then*

$$(5.3) \quad \int_0^r \frac{m(t, \varphi, \theta)}{t} dt = \frac{1}{2\pi} \int_0^r \frac{d}{d\theta} J_f^t(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^r \frac{1}{t} \int_0^t \frac{d}{ds} \arg f(se^{i\varphi}) ds dt + \frac{1}{2\pi} \int_\varphi^\theta \ln |f(re^{i\omega})| d\omega.$$

Proof. Using the argument principle and the Cauchy-Riemann equations just as in [11, Chapter 3, section 2] we see that

$$2\pi m(r', \varphi, \theta) = \int_0^{r'} \frac{\partial}{\partial t} \arg f(te^{i\varphi}) dt + \int_0^{r'} \frac{1}{t} \frac{\partial}{\partial \theta} \ln |f(te^{i\theta})| dt + r' \int_\varphi^\theta \frac{\partial}{\partial r'} \ln |f(r'e^{i\omega})| d\omega$$

when there are no zeros on the boundary of the sector. As in [11], by dividing by $2\pi r'$ and integrating from 0 to r in r' we obtain the lemma. \square

We note that $|s_V(0)| = 1$, since $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 5.2. *Suppose $V \in L_{\text{comp}}^\infty(\mathbb{R}^d)$. Then for $0 < \theta < \pi$*

$$\int_0^\theta N_V(r, \pi, \theta' + \pi) d\theta' = \frac{1}{2\pi} \int_0^r J_{s_V}^t(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^\theta \int_0^{\theta'} \ln |s_V(re^{i\omega})| d\omega d\theta' + O(r^{d-1})$$

as $r \rightarrow \infty$. The error can be bounded by $c\langle r^{d-1} \rangle$ where the constant depends only on $\|V\|_\infty$, the support of V , and d .

Proof. Recall that with at most a finite number of exceptions λ_0 is a pole of $R_V(\lambda)$ if and only if $-\lambda_0$ is a zero of $s_V(\lambda)$, and the multiplicities coincide. As in the proof of Proposition 1.1, if $s_V(\lambda)$ has poles $\lambda_1, \dots, \lambda_m$ in the closed upper half plane we introduce the function

$$f(\lambda) = \frac{(\lambda - \lambda_1) \cdots (\lambda - \lambda_m)}{(\lambda + \lambda_1) \cdots (\lambda + \lambda_m)} s_V(\lambda)$$

which is holomorphic in the closed upper half plane. The poles of s_V in the closed upper half plane correspond to eigenvalues and the number of such poles can be bounded by a constant depending on d , $\|V\|_\infty$, and the support of V . Note that f

has no zeros on the real line. Moreover, $\ln |f(re^{i\theta})| = \ln |s_V(re^{i\theta})| + O(1)$ for $r \rightarrow \infty$, $0 \leq \theta \leq \pi$ and that s_V and f have all but finitely many of the same zeros.

Choose $0 < M < \infty$ so that $s_V(\lambda)$ has no zeros in the upper half plane with norm less than or equal to M . This constant M can be chosen to depend only on $\|V\|_\infty$, the support of V , and d . Now by using the relationship between the poles of $R_V(\lambda)$ and the zeros of $s_V = \det S_V$, the relationships between f and s_V just mentioned, and applying Lemma 5.1 to f we see that for $r > M$, $0 < \theta' < \pi$,

$$(5.4) \quad N_V(t, \pi, \theta' + \pi) = \frac{1}{2\pi} \int_M^r \frac{\partial}{\partial \theta'} J_{s_V}^t(\theta') \frac{dt}{t} + \frac{1}{2\pi} \int_M^r \frac{1}{t} \int_M^t \frac{d}{dt'} \arg s_V(t') dt' dt \\ + \frac{1}{2\pi} \int_0^{\theta'} \ln |s_V(re^{i\omega})| d\omega + O((\ln r)^2)$$

if f has no zeros with argument θ' and norm not exceeding r . Here we are using that $\int_0^M \frac{\partial}{\partial \theta'} J_f^t(\theta') \frac{dt}{t} = O(1)$ and

$$\int_0^r \frac{1}{t} \int_0^t \frac{d}{dt'} \arg f(t') dt' dt \\ = \int_M^r \frac{1}{t} \int_M^t \frac{d}{dt'} \arg f(t') dt' dt + \int_M^r \frac{1}{t} \int_0^M \frac{d}{dt'} \arg f(t') dt' dt + \int_0^M \frac{1}{t} \int_0^t \frac{d}{dt'} \arg f(t') dt' dt.$$

But $\int_M^r \frac{1}{t} \int_0^M \frac{d}{dt'} \arg f(t') dt' dt = O(\ln r)$ and $\int_0^M \frac{1}{t} \int_0^t \frac{d}{dt'} \arg f(t') dt' dt = O(1)$. Additionally, for $t \rightarrow \infty$, $\frac{d}{dt} \arg f(t) = \frac{d}{dt} \arg s_V(t) + O(1/t)$. These remainders can be bounded using constants depending only on $\|V\|_\infty$, $\text{supp } V$, and d .

Notice that for fixed value of $r > M$ there are only finitely many values of θ' with s_V having a zero with argument θ' and norm at most r . We integrate (5.4) in θ' from 0 to θ and, as in the proof of Jensen's equality, use the fact that both sides of the equation below are continuous functions of θ , to get

$$\int_0^\theta N_V(r, \pi, \theta' + \pi) d\theta' = \frac{1}{2\pi} \int_M^r J_{s_V}^t(\theta) \frac{dt}{t} - \frac{1}{2\pi} \int_M^r J_{s_V}^t(0) \frac{dt}{t} \\ + \frac{\theta}{2\pi} \int_M^r \frac{1}{t} \int_M^t \frac{d}{dt'} \arg s_V(t') dt' dt + \frac{1}{2\pi} \int_0^\theta \int_0^{\theta'} \ln |s_V(re^{i\omega})| d\omega d\theta' + O((\ln r)^2).$$

The bounds of Lemma 3.1 and (3.1) mean that as $r \rightarrow \infty$

$$\frac{1}{2\pi} \int_M^r J_{s_V}^t(0) \frac{dt}{t} = O(r^{d-1})$$

and

$$\frac{\theta}{2\pi} \int_M^r \frac{1}{t} \int_M^t \frac{d}{dt'} \arg s_V(t') dt' dt = O(r^{d-1})$$

where the bounds can be made uniform in V with support contained in a fixed compact set and $\|V\|_\infty$ bounded. Moreover, we note that $\int_0^M J_{s_V}^t(\theta) \frac{dt}{t} = O(1)$. \square

We shall need some notation for the results which follow. Let $\Omega \subset \mathbb{C}^{d'}$ be an open set containing a point z_0 . For $\rho > 0$ small enough that $B(z_0, \rho) \subset \Omega$ we define Ω_ρ to be the connected component of $\{z \in \Omega : \text{dist}(z, \Omega^c) > \rho\}$ which contains z_0 .

Proposition 5.3. *Let V , z_0 , Ω satisfy the assumptions of Theorem 1.2, let $\rho > 0$ be small enough that $B(z_0, 2\rho) \subset \Omega$, and let Ω_ρ be as defined above. Then for $z \in \Omega_{2\rho}$, $0 < \theta < \pi$,*

$$\begin{aligned} \Psi(z, r, \rho) &\stackrel{\text{def}}{=} \frac{1}{\text{vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_0^\theta N_{V(z')}(r, \pi, \theta' + \pi) d\theta' d\mathcal{L}(z') \\ &= \frac{1}{2\pi} a^d r^d \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) + o(r^d) \end{aligned}$$

as $r \rightarrow \infty$.

Proof. First note that since $0 \leq dN_{V(z)}(z, \pi, \theta + \pi) \leq c_d r^d a^d + o(r^d)$, and the bound is uniform on compact sets of z , we get that holding ρ fixed, $r^{-d} \Psi(\bullet, r, \rho)$ is a family uniformly continuous in z for z in compact sets of $\overline{\Omega}_{2\rho}$.

We shall use Lemma 5.2. Note that by Stefanov's results recalled in Theorem 3.2, for large r

$$\frac{1}{2\pi} \int_0^r J_{s_{V(z)}}^t(\theta) \frac{dt}{t} \leq \frac{1}{2\pi} \frac{1}{d^2} h_d(\theta) a^d r^d + o(r^d)$$

where the term $o(r^d)$ can be bounded uniformly in z in compact sets of $\overline{\Omega}_\rho$. Recall that this is a statement about large r behavior, and holds even if $s_V(z)$ has poles in the upper half plane, since it has at most finitely many. By the same argument, for large r

$$\int_0^\theta \int_0^{\theta'} \ln |s_{V(z)}(r e^{i\omega})| d\omega d\theta' \leq \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' a^d r^d + o(r^d).$$

Using Lemma 5.2, we find that

$$\begin{aligned} \Psi(z, r, \rho) &= \frac{1}{2\pi \text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_0^r J_{s_{V(z')}}^t(\theta) \frac{dt}{t} d\mathcal{L}(z') \\ &\quad + \frac{1}{2\pi \text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_0^\theta \int_0^{\theta'} \ln |s_{V(z')}(r e^{i\omega})| d\omega d\theta' d\mathcal{L}(z') + O(r^{d-1}). \end{aligned}$$

Let $M = 2\alpha_d \max_{z \in \overline{\Omega}_\rho} \|V(z)\|_\infty$ and set, for $r > M$,

$$\begin{aligned} \Psi_1(z, r, \rho) &= \frac{1}{2\pi \text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_M^r J_{s_{V(z')}}^t(\theta) \frac{dt}{t} d\mathcal{L}(z') \\ &\quad + \frac{1}{2\pi \text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} \int_0^\theta \int_0^{\theta'} \ln |s_{V(z')}(r e^{i\omega})| d\omega d\theta' d\mathcal{L}(z') \end{aligned}$$

and note that

$$\Psi(z, r, \rho) = \Psi_1(z, r, \rho) + O(r^{d-1}).$$

By the bounds above,

$$(5.5) \quad \Psi_1(z, r, \rho) \leq \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d r^d + o(r^d).$$

Using [10, Proposition I.14] and the fact that $\ln |s_{V(z)}(\lambda)|$ is a plurisubharmonic function of $z \in \Omega$ when $|\lambda| > 2\alpha_d \|V(z)\|_\infty$ and λ lies in the upper half plane, we see that $\Psi_1(z, r, \rho)$ is a plurisubharmonic function of $z \in \Omega_{2\rho}$. Since by Proposition 2.2 $s_{V(z_0)}(\lambda)$ is of completely regular growth in $0 < \arg \lambda < \pi$, using Lemma 5.2 and [11, Chapter III, Sec. 2, Lemma 2],

$$\lim_{r \rightarrow \infty} r^{-d} \int_0^\theta N_{V(z_0)}(r, \pi, \theta' + \pi) d\theta' = \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d.$$

By the most basic property of plurisubharmonic functions,

$$\Psi_1(z_0, r, \rho) \geq \frac{1}{2\pi} \int_M^r J_{s_{V(z_0)}}^t(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^\theta \int_0^{\theta'} \ln |s_{V(z_0)}(re^{i\omega})| d\omega d\theta'.$$

But the right hand side of this equation is $\int_0^\theta N_{V(z_0)}(r, \pi, \theta' + \pi) d\theta' + O(r^{d-1})$, so we see that

$$\liminf_{r \rightarrow \infty} r^{-d} \Psi_1(z_0, r, \rho) \geq \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d.$$

Combining this with (5.5), we find

$$(5.6) \quad \lim_{r \rightarrow \infty} r^{-d} \Psi_1(z_0, r, \rho) = \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d.$$

Using this and the upper bound (5.5) on Ψ_1 , since Ψ_1 is plurisubharmonic in z it follows from [10, Proposition 1.39] (recalled here in Proposition 2.3) that for any sequence $\{r_j\}$, $r_j \rightarrow \infty$ there is a pluripolar set $E \subset \Omega_\rho$ (which may depend on the sequence) so that

$$\limsup_{j \rightarrow \infty} r_j^{-d} \Psi_1(z, r_j, \rho) = \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d$$

for all $z \in \Omega_\rho \setminus E$. Since $\lim_{r \rightarrow \infty} r^{-d} (\Psi_1(z, r, \rho) - \Psi(z, r, \rho)) = 0$, the same conclusion holds for Ψ in place of Ψ_1 .

Suppose there is some $z_1 \in \Omega_\rho$ and some sequence $r_j \rightarrow \infty$ so that

$$\lim_{j \rightarrow \infty} r_j^{-d} \Psi(z_1, r_j, \rho) < \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d.$$

Then, using the uniform continuity of $r^{-d}\Psi(z, r, \rho)$ in z , we find there must be an $\epsilon > 0$ so that

$$\limsup_{j \rightarrow \infty} r_j^{-d}\Psi(z, r_j, \rho) < \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d$$

for all $z \in B(z_1, \epsilon)$. But since $B(z_1, \epsilon)$ is not contained in a pluripolar set, we have a contradiction. Thus

$$\lim_{r \rightarrow \infty} r^{-d}\Psi(z, r, \rho) = \frac{1}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) a^d$$

for all $z \in \Omega_\rho$. □

The following lemma will be used to remove the need to average in θ as in Proposition 5.3.

Lemma 5.4. *Let $M(r, \theta)$ be a function so that for any fixed positive $r_0 > C_0$, $M(r_0, \theta)$ is a non-decreasing function of θ , and suppose*

$$\lim_{r \rightarrow \infty} r^{-d} \int_0^\theta M(r, \theta') d\theta' = \alpha(\theta)$$

for $\theta_1 < \theta < \theta_2$. Then if α is differentiable at θ , then

$$\lim_{r \rightarrow \infty} r^{-d} M(r, \theta) = \alpha'(\theta).$$

Proof. Let $\epsilon > 0$. Then, since $M(r, \theta)$ is non-decreasing in θ ,

$$\int_0^{\theta+\epsilon} M(r, \theta') d\theta' - \int_0^\theta M(r, \theta') d\theta' \geq \epsilon M(r, \theta)$$

which, under rearrangement, yields

$$r^{-d} M(r, \theta) \leq r^{-d} \frac{\int_0^{\theta+\epsilon} M(r, \theta') d\theta' - \int_0^\theta M(r, \theta') d\theta'}{\epsilon}.$$

Thus

$$\limsup_{r \rightarrow \infty} r^{-d} M(r, \theta) \leq \frac{\alpha(\theta + \epsilon) - \alpha(\theta)}{\epsilon}.$$

Likewise, we find

$$\liminf_{r \rightarrow \infty} r^{-d} M(r, \theta) \geq \frac{\alpha(\theta) - \alpha(\theta - \epsilon)}{\epsilon}.$$

Since both these equalities must hold for all $\epsilon > 0$, the lemma follows from the assumption that α is differentiable at θ . □

The following proposition follows from Proposition 5.3, but is stronger as it does not require averaging in the θ' variables.

Proposition 5.5. *Let V , z_0 , Ω satisfy the assumptions of Theorem 1.2, and $\rho > 0$, Ω_ρ be as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, $0 < \theta < \pi$, as $r \rightarrow \infty$*

$$\begin{aligned} \frac{1}{\text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} N_{V(z')}(r, \pi, \theta + \pi) d\mathcal{L}(z') \\ = \frac{1}{2\pi} a^d r^d \left(\frac{1}{d^2} h'_d(\theta) + \int_0^\theta h_d(\omega) d\omega \right) + o(r^d). \end{aligned}$$

Proof. This follows from applying Lemmas 5.4 and 3.3 to the results of Proposition 5.3. \square

Proposition 5.5 does not give results for the counting function for all the resonances (note that we cannot have $\theta = \pi$). The following fills this gap.

Proposition 5.6. *Let V , z_0 , Ω satisfy the assumptions of Theorem 1.2, and $\rho > 0$, Ω_ρ as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, as $r \rightarrow \infty$*

$$\frac{1}{\text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} N_{V(z')}(r) d\mathcal{L}(z') = \frac{1}{2\pi} a^d r^d \int_0^\theta h_d(\omega) d\omega + o(r^d).$$

Proof. The proof of this is very similar to that of Proposition 5.3. In fact, the main difference is the use of (2.2), which together with Lemma 3.1 and (3.1) gives us by handling possible poles in the upper half plane using a method similar to the proof of Lemma 5.2,

$$\frac{1}{\text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} N_{V(z')}(r) d\mathcal{L}(z') = \Psi_1(z, r, \rho) + O(r^{d-1})$$

where

$$\Psi_1(z, r, \rho) = \frac{1}{\text{Vol}(B(z, \rho))} \frac{1}{2\pi} \int_{z' \in B(z, \rho)} \int_0^\pi \ln |s_{V(z')}(r e^{i\theta})| d\theta d\mathcal{L}(z').$$

Using that Ψ_1 is plurisubharmonic in z , the proof now follows just as in Proposition 5.3. \square

The following proposition is much like Propositions 5.5 and 5.6, but eliminates the average in the r variable.

Proposition 5.7. *Let V , Ω , z_0 satisfy the conditions of Theorem 1.2, and let ρ , Ω_ρ be as in Proposition 5.3. Then for $0 < \theta < \pi$, $z \in \Omega_{2\rho}$,*

$$\frac{1}{\text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} n_{V(z')}(r, \pi, \theta + \pi) d\mathcal{L}(z') = \frac{a^d r^d}{2\pi} \left(\frac{1}{d} h'_d(\theta) + d \int_0^\theta h_d(\theta) d\theta \right) + o(r^d)$$

and

$$\frac{1}{\text{Vol}(B(z, \rho))} \int_{z' \in B(z, \rho)} n_{V(z')}(r) d\mathcal{L}(z') = \frac{d}{2\pi} a^d r^d \int_0^\pi h_d(\theta) d\theta + o(r^d)$$

as $r \rightarrow \infty$.

Proof. This proof follows from Propositions 5.5 and 5.6, using, in addition, a result like that of [18, Lemma 1] or Proposition 5.4. \square

Proof of Theorem 1.3. Let $M = \max(1 + |\psi(z)|)$, and for $\rho > 0$ small enough that $B(z_0, \rho) \subset \Omega$, set Ω_ρ to be the connected component of $\{z \in \Omega : \text{dist}(z, \Omega^c) > \rho\}$ which contains z_0 . Given $\epsilon > 0$, choose $\rho > 0$ such that $B(z_0, 2\rho) \subset \Omega$ and so that

$$(5.7) \quad \text{vol}(\text{supp } \psi \cap (\Omega \setminus \Omega_{2\rho})) < \frac{\epsilon}{10Me(c_d a^d + 1)}.$$

Since ψ is continuous with compact support, we can find a $\delta_1 > 0$, $\delta_1 < \rho$ so that if $|z - z'| < \delta_1$, then $|\psi(z) - \psi(z')| < \frac{\epsilon}{10e(1 + \text{vol supp } \psi)(a^d c_d + 1)}$. We may find a finite number J of disjoint balls $B(z_j, \epsilon_j)$ so that $\epsilon_j < \delta_1$, $z_j \subset \Omega_{2\rho}$, and

$$\text{vol}(\text{supp } \psi \setminus (\cup_1^J B(z_j, \epsilon_j))) + \text{vol}(\cup_1^J B(z_j, \epsilon_j) \setminus \text{supp } \psi) < \frac{\epsilon}{4Me(a^d c_d + 1)}.$$

Let $\pi \leq \varphi' \leq \theta' \leq 2\pi$. Now

$$\begin{aligned} & \int \psi(z) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z) \\ &= \sum_{j=1}^J \int_{B(z_j, \epsilon_j)} \psi(z) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z) + \int_{\text{supp } \psi \setminus (\cup B(z_j, \epsilon_j))} \psi(z) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z). \end{aligned}$$

We will use that the bound (1.1) implies that $n_V(z) \leq e c_d a^d r^d + o(r^d)$. By our choice of $B(z_j, \epsilon_j)$,

$$\left| \int_{\text{supp } \psi \setminus (\cup B(z_j, \epsilon_j))} \psi(z) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z) \right| \leq \frac{\epsilon}{4} (r^d + o(r^d)).$$

By our choice of δ_1 and the assumption that $\epsilon_j < \delta_1$, we have

$$\begin{aligned} & \left| \sum_{j=1}^J \int_{B(z_j, \epsilon_j)} \psi(z) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z) - \sum_{j=1}^J \int_{B(z_j, \epsilon_j)} \psi(z_j) n_{V(z)}(r, \varphi', \theta') d\mathcal{L}(z) \right| \\ & \leq \frac{\epsilon}{5} (r^d + o(r^d)). \end{aligned}$$

By Proposition 5.7, if $0 < \theta < \pi$,

$$\begin{aligned} & \sum_{j=1}^J \int_{B(z_j, \epsilon_j)} \psi(z_j) n_{V(z)}(r, \pi, \pi + \theta) d\mathcal{L}(z) \\ &= \left(\sum_{j=1}^J \psi(z_j) \text{vol}(B(z_j, \epsilon_j)) \right) \frac{1}{2\pi} a^d r^d \left(\frac{1}{d} h'_d(\theta) + d \int_0^\theta h_d(\omega) d\omega \right) + o(r^d), \end{aligned}$$

and

$$\sum_{j=1}^J \int_{B(z_j, \epsilon_j)} \psi(z_j) n_{V(z)}(r) d\mathcal{L}(z) = \left(\sum_{j=1}^J \psi(z_j) \text{vol}(B(z_j, \epsilon_j)) \right) \frac{d}{2\pi} a^d r^d \int_0^\pi h_d(\omega) d\omega + o(r^d).$$

Again using our choice of δ_1 , z_j , and ϵ_j , we have

$$\left| \sum_{j=1}^J \psi(z_j) \text{vol}(B(z_j, \epsilon_j)) - \int \psi(z) d\mathcal{L}(z) \right| < \frac{2\epsilon}{5(c_d a^d + 1)}.$$

Thus we have shown that given $\epsilon > 0$, if $0 < \theta < \pi$,

$$(5.8) \quad \left| \int \psi(z) n_{V(z)}(r, \pi, \theta + \pi) d\mathcal{L}(z) - \frac{a^d r^d}{2\pi} \int \psi(z) d\mathcal{L}(z) \left(\frac{1}{d} h'_d(\theta) + d \int_0^\theta h_d(\omega) d\omega \right) \right| \leq \epsilon r^d + o(r^d)$$

and

$$(5.9) \quad \left| \int \psi(z) n_{V(z)}(r) d\mathcal{L}(z) - c_d a^d r^d \int \psi(z) d\mathcal{L}(z) \right| \leq \epsilon r^d + o(r^d).$$

Thus we have proved the first and third statements of the theorem. The second statement of the theorem follows from the other two.

6. PROOF OF THEOREM 1.2

This proof uses some ideas similar to those used in the proofs of Propositions 5.3 and 5.6. In fact, because the proofs are so similar we shall only give an outline.

Note that by (2.2), (3.1), and Lemma 3.1, using an argument similar to the proofs of Lemma 5.2 and Proposition 5.3,

$$N_{V(z)}(r) = \Psi(z, r) + o(r^{d-1})$$

where

$$\Psi(z, r) = \frac{1}{2\pi} \int_0^\pi \ln |s_{V(z)}(r e^{i\theta})| d\theta$$

is, for fixed (large) r a plurisubharmonic function of $z \in \tilde{\Omega} \Subset \Omega$. Since

$$\limsup_{r \rightarrow \infty} r^{-d} \Psi(z, r) \leq \frac{a^d}{2\pi} \int_0^\pi h_d(\theta) d\theta$$

and this maximum is achieved at $z = z_0 \in \Omega$, we get the first part of the Theorem by applying [10, Proposition 1.39], recalled in Proposition 2.3.

To obtain the second part, note that as in the proof of Proposition 5.3, for $0 < \theta < \pi$,

$$\int_0^\theta N_{V(z)}(r, \pi, \theta' + \pi) d\theta' = \Psi_2(z, r, \theta) + o(r^d)$$

where

$$\Psi_2(z, r, \theta) = \frac{1}{2\pi} \int_M^r J_{s_V(z)}^t(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^\theta \int_0^{\theta'} \ln |s_V(z)(re^{i\omega})| d\omega d\theta'.$$

Since this is a plurisubharmonic function of $z \in \tilde{\Omega}$, $\tilde{\Omega} \Subset \Omega$, if M is chosen so that $M \geq 2\alpha_d \max_{z \in \tilde{\Omega}} \|V\|_\infty$, a similar argument as in the proof of Proposition 5.3 shows that there exists a pluripolar set $E_\theta \subset \Omega$ so that

$$2\pi \limsup_{r \rightarrow \infty} r^{-d} \Psi_2(z, r, \theta) = a^d \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right)$$

for all $z \in \Omega \setminus E_\theta$. Again, we use that this equality holds when $z = z_0$. Note that if the second part of the theorem can be proved for a small θ_0 , it is proved for all θ with $\theta \geq \theta_0$. Thus, it is most interesting for small θ . Choose $\theta > 0$ sufficiently small that $h_d(\theta) \geq \theta h'_d(0+)/2$, where we denote $\lim_{\epsilon \downarrow 0} h'_d(\epsilon) = h'_d(0+)$. Note that $h_d(\theta) \geq 0$. Now, if

$$\begin{aligned} \limsup_{r \rightarrow \infty} r^{-d} \int_0^\theta N_V(r, \pi, \pi + \theta') d\theta' &= \frac{a^d}{2\pi} \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega) d\omega d\theta' \right) \\ &\geq \frac{a^d}{4\pi d^2} h'_d(0+) \theta, \end{aligned}$$

then since $N_V(r, \pi, \pi) = O(1)$, we must have

$$\limsup_{r \rightarrow \infty} r^{-d} N_V(r, \pi, \pi + \theta) \geq \frac{a^d}{4\pi d^2} h'_d(0+).$$

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